for short axial wavelengths become parallel to the sector line ($\beta_1 = 0$) with the least slope. For shells with $\mu = 1$, $\rho = 2$, the frequency lines of the third and fourth axisymmetric nontorsional modes become parallel to the frequency line of the Rayleigh waves in the material of the outer shell, prior to becoming parallel to the sector line $\beta_2 = 0$ (Fig. 9), whereas, in the case of shells with $\mu = 2$, $\rho = 1$, they become parallel to the frequency line of the Rayleigh waves in the material of the inner shell prior to becoming parallel to the sector line $\beta_1 = 0$.

The frequency lines of the fourth, fifth and sixth torsional modes and the fifth and sixth nontorsional modes for shells with $\mu = 1$, $\rho = 2$, become parallel to the sector line $\beta_2 = 0$ before, for higher values of ζ not included in the figures, becoming parallel to the sector line $\beta_1 = 0$. Analogous behavior is observed for shells with $\mu = 2$, $\rho = 1$.

The behavior of the frequency lines of the flexural modes ($n \geq 1$,) is similar to that of the axisymmetric modes, except that they do not exhibit the tendency of becoming parallel to the sector line $\beta_i = 0$, with the greatest slope. For $n = 1$, for large axial wavelengths the first mode is essentially a uniform translation of the entire cross section, the second mode involves essentially longitudinal shear motion, whereas the third mode is associated with predominantly radial motion (breathing). The three lowest flexural modes are those contained in the bending shell theories, wherein the radial component of the displacement is assumed constant across the thickness of the shell and the tangential and axial components are assumed to vary linearly with the thickness coordinate.

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A Study of the Buckling of Laminated Composite Plates

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Buckling loads are determined for approximately square, simply supported plates of the following materials: aluminum, steel, glass fiber reinforced composites and graphite fiber reinforced composites. The plates are subjected to a uniformly distributed compressive edge load in one direction only. Buckling loads are determined from experimental load vs transverse displacement data using a Southwell Plot. Analytical buckling loads for elastically balanced plates are determined from classical plate theory. For plates with bending-membrane coupling effects, the reduced flexural stiffness method is utilized to find eigenvalue upper bounds. The effectiveness and consequences of using the reduced flexural stiffness method are examined.

Nomenclature

a,b	= planform dimensions of the plate
A,B,D	= element matrices of the constituitive equation
A^*,B^*,D^*	= element matrices of the partial inverse form of the constituitive equations
(e)	= vector of strain components (e_x,e_y,e_{xy})
(κ)	= vector of curvature components $(\kappa_x,\kappa_y,\kappa_{xz})$
L_1,L_2,L_3,L_4	= differential operators
m,n	= buckling wave numbers
(M)	= vector of mement resultants (M_x,M_y,M_{xy})
(N)	= vector of force resultants (N_x,N_y,N_{xy})
P	= total axial load
u,v,w	= displacements
V	= strain energy of deformation
x,y,z	= coordinate variables
Δ	= maximum transverse displacement
θ	= orientation of the fibers in a single ply referenced to the x axis (Fig. 1)
Π	= generalized potential function
ϕ	= airy stress function

Subscripts

cr	= critical or buckling value
o	= prebuckling
,	= differentiation with respect to the variables which follow the comma

Superscripts

$\sim$	= quantities associated with buckling
—	= quantities imposed at the boundaries

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Introduction

THE use of high performance composite materials has intensified the demand for the development of analytical methods for anisotropic structures. To date, composite materials have been used extensively for tensile loaded structures such as pressure vessels. Recently, composite materials have been suggested for compression loaded structures such as wing surfaces, fuselage sections, and missile skins. The analytical tools available for composite material structures are based extensively on the classical literature. The experimental data available for substantiation is limited. More recently, analytical tools, in the form of generalized computer programs, have become available. However, computational costs discourage their use in extensive preliminary design. Frequently a designer will sacrifice analytical accuracy for speed and economy in design.

The application of high performance composite materials to flight vehicle hardware intensifies the need for accurate analytical tools, easily adapted to engineering designs.

The quest for minimum weight design of compression load structures of high performance composite materials leads to a minimum gage problem similar to the situation with conventional metallic structures. An additional complication arises, however, when a minimum gage requirement is imposed on a composite structure laminated from plies of a fixed thickness; for example, the number of combinations of ply arrangements for four plies is extremely limited. In addition, some of the combinations are elastically unsymmetrical about the midsurface so that upon application of a membrane load the plate will deflect in a bending mode. Buckling is therefore not characterized by a bifurcation response but rather a nonlinear large displacement phenomenon. Initial imperfections play a similar role in the buckling response of elastically symmetric plates, but it has been shown[1] that such imperfections do not significantly affect the "buckling load."

Several years ago the authors became involved in the study of the buckling of laminated plates. A test fixture was designed and fabricated, and several metallic plates were tested to verify the experimental equipment and procedures.[2] The buckling loads were deduced from the load versus transverse displacement data using a Southwell plot.[3] Next several laminated plates were fabricated from single plies of prepreg fiberglass and "Thornel."‡ Initially only plates with plies laminated symmetrically with respect to the midsurface were considered. The correlation of the buckling loads with the analytical predictions of orthotropic plate theory of Lekhnitskii[4] was well within accepted experimental error. It became obvious early in the program that plates with bending-membrane coupling were of interest to designers but they had no means of estimating the effects of such coupling. Several unbalanced plates were tested, and it was observed that the buckling loads were substantially below those predicted by classical plate theory ignoring the coupling terms. At the suggestion of Chamis,[5] the reduced flexural stiffness of Reissner and Stavsky[6] were used to calculate the buckling load with classical theory. The correlation of the experimental data with the approximate buckling load was substantially improved over the results of classical theory completely ignoring coupling. The results were summarized and presented at the AIAA/ASME 10th Structures, Structural Dynamics and Materials Conference, New Orleans, La., April 14–16, 1969. In the light of more recent work,[7–14] the results presented earlier are reviewed, and the full meaning of bending-membrane coupling and the reduced flexural stiffness approximation for the buckling analysis of laminated plates is examined.

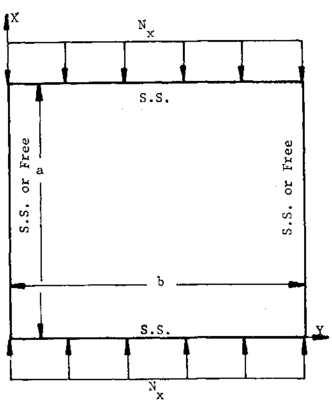

Fig. 1 Diagram of a typical test specimen with applied load and boundary conditions.

‡ A product of Union Carbide Corporation.

Experimental Program

The buckling of laminated, fiber reinforced composite plates has been investigated experimentally in order to determine the applicability of classical buckling theories and the effects of bending-membrane coupling. Square, or nearly square, plates were subjected to a uniform compressive edge load in one direction only, and the following boundary conditions were observed: 1) loaded edges simply supported, unloaded edges free, and 2) all four edges simply supported, see Fig. 1. On the loaded edges the simply supported boundary conditions were achieved by attaching wedge shaped bearing points to prevent crushing of the edges of the brittle specimens. The wedges added approximately 0.5 in. to the length of the 10 in. × 10 in. specimen. This will introduce approximately a 1% error in the calculation of the buckling load. On the unloaded edges, the simply supported boundary condition was achieved by confining the plate between rollers; therefore the unloaded edges are free to expand in the plane of the plate. The details of these support conditions are shown in Figs. 2–4. For the case of free unloaded edges, the roller supports and side rails were removed to allow the free displacement of the plate as a wide column.

The test fixture, described in detail in Ref. 2 and shown in Figs. 2–5, is unusual in that it imposes a uniform load rather than the usual uniform displacement. The uniform distribu-

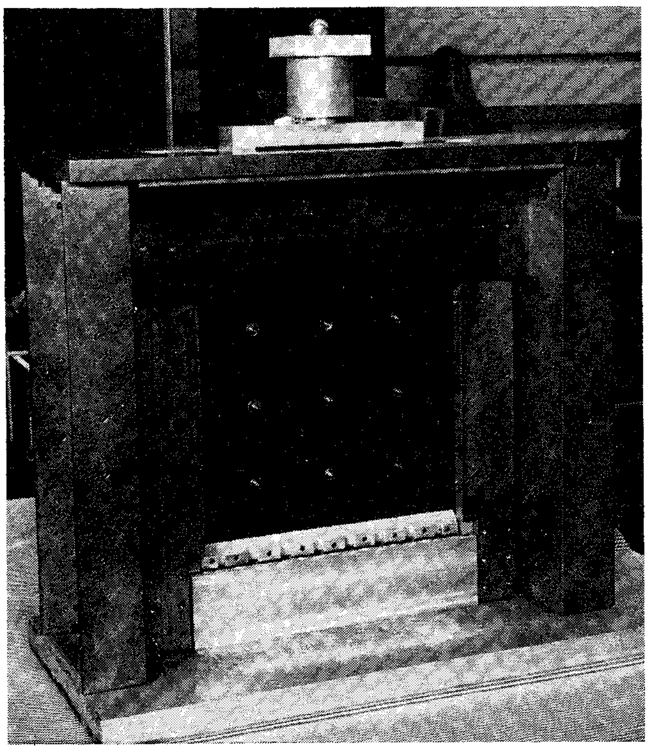

Fig. 2 Photograph of test fixture.

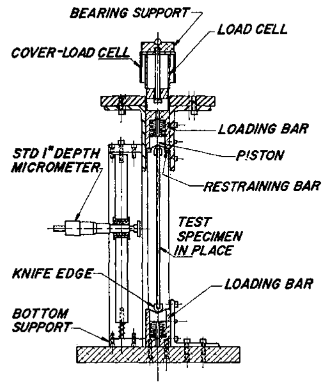

Fig. 3 Side view of text fixture.

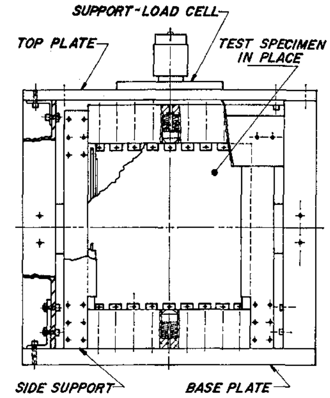

Fig. 4 Front view of text fixture.

tion of the load was maintained through a soft loading head so that nonuniform distributions in the end displacement would not induce a nonuniform stress into the specimen. Each of the bearing wedges is fitted into a vee-grooved, spring supported piston. The springs are selected to match the axial stiffness of the plate. As the load is applied, the springs displace relative to each other maintaining an approximate uniform distribution of load. The deformation of the lower set of springs introduces a rigid body motion necessitating the floating of the vertical side supports. The hardened steel rollers providing the simply supported boundary conditions on the sides were mounted in cages which slide in the fixed brass side rails. The total load is determined from strain gage readings on a thin walled cylindrically shaped load cell.

As the plate is loaded, lateral deflections are measured by means of a micrometer that makes an electrical contact with the plate. Deflections are taken at the point of maximum deflection to obtain the load-deflection curve and at a number of additional points to define the shape of the deflected surface.

It was also necessary to accurately shift the displacement transducer vertically to maintain contact at the point of maximum displacement. Once the load-deflection curve is obtained for a plate, the Southwell plot technique[3] is employed to determine the "buckling load." Tests were run on thirty-two plates encompassing the following materials: aluminum, steel, glass fiber, epoxy prepreg, and graphite fiber reinforced composite.

The aluminum and steel specimens were rough cut with a shear from flat sheet stock and milled to a square configuration maintaining tolerances of ± 0.002 in. The fiberglass plates were fabricated from 3-M's, Type 1002 prepreg and cured in a hot press at 320°F for approximately 35 min. The edges of the plate were constrained in the mold to achieve good fiber alignment. A bleader cloth was placed over the surfaces of the plate to remove the excess resin. Typical fiber volume contents ranged from 55% to 67% by volume as determined optically. The "Thornel" plates were fabricated by Union Carbide Corporation, Parma Technical Center. The elastically unbalanced plates were fabricated by first curing the balanced sublaminates and subsequently bonding them with a room temperature curing epoxy. This procedure prevents the warping of the unsymmetrical sections due to differential temperature and shrinkage stresses. All composite plates were rough cut to size on a saw and ground to a square configuration with a water cooled cut-off wheel maintaining tolerances of ± 0.002 in.

Stiffness Properties for Fiber Composite Plates

The buckling load calculations all must ultimately rely on the prediction of the elastic stiffnesses of the plate. Prediction of these stiffnesses first requires the accurate determination of the elastic properties of a single ply of the plate. The ply properties were determined by a number of methods for the different materials, and can be briefly described as follows: 1) ply properties determined by Whitney's micromechanic method[15] (fiberglass and graphite plates), and 2) ply properties determined by semiempirical method[16] (graphite plates). The resulting stiffnesses for all of the plates tested along with the assumed material properties can be found in Ref. 2.

Analysis of Laminated Plates

Classical orthotropic buckling theory[4] can be applied to isotropic or to orthotropic plates with the material axes aligned with the structural axes of the plate and with the plies oriented symmetrically about the midplane. For the case where the principal axes of the material do not coincide with the edges of the rectangular plate, in-plane shear coupling effects are introduced. Ashton and Waddoups[4] utilized the Ritz technique to find the buckling loads for rectangular plates with various boundary conditions. The displacement modes were approximated by products of beam functions. The moment boundary condition is not exactly satisfied for simply supported or free edges, but the results were found to be acceptable. Ashton[8] identified the analogy between skewed isotropic plates and anisotropic plates. This allows the direct adaptation of solutions for skewed isotropic plates to rectangular anisotropic plates for which D_{16} and D_{26} are not zero. Wang[9] has observed that a separation of variables cannot be utilized in seeking an exact solution for simply supported anisotropic plates. The exact satisfaction of the moment boundary condition precludes the use of a separation of variables, if the Love-Kirchoff hypothesis is accepted.

Reissner and Stavsky[6] were the first to identify the full complexity associated with the analysis of an unbalanced angle-ply plate. The analysis for the response of a $\pm\ \theta$ two-layer fiber composite plate was formulated in terms of a stress function, ϕ and the out of plane displacement w. They defined the reduced flexural stiffness terms $(D - BA^{-1}B)$ and proposed an approximate solution method neglecting the coupling terms: $BA^{-1}N$ in the transverse equilibrium equation and $-A^{-1}B\kappa$ in the membrane compatibility equation. This simplification allows the direct adaptation of classical analysis methods to plates with bending-membrane coupling.

Whitney and Leissa[10] presented the analysis for cross plied and angle plies laminated composite plates in terms of a stress function ϕ and the out-of-plane displacement w. They ex-

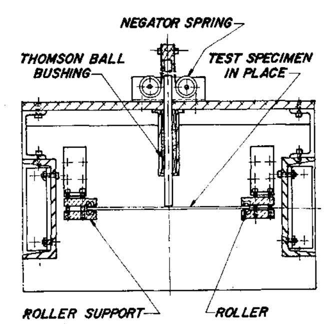

Fig. 5 Top view of test fixture.

tended the work of Reissner and Stavsky,[6] by presenting solutions for the transverse vibration and eigenvalue buckling analysis of angle ply composite plates. They found the exact solutions for the buckling load of angle-ply composites, including the effects of coupling, assuming that the prebuckling response for an angle-ply composite is a pure membrane response. Results for the buckling of cross-ply composites were not presented. They also presented the exact solution for an arbitrary lateral load using a Green[17] type of solution for simply supported boundary conditions.

Whitney[11] and Whitney and Leissa[12] have formulated the analysis of cross-plied and angle-plied composite plates in terms of three equilibrium equations written in terms of displacement variables u, v, and w. The stability analysis[12] of simply supported angle-ply plates is cast in terms of the displacements associated with buckling and the loads prior to buckling. The results obtained are in agreement with those presented by Whitney and Leissa.[10] The buckling analysis for simply supported cross plates is again omitted.

Ashton and Love[13] presented the results of an experimental study on the buckling of clamped plates. The results include the buckling loads for 24 metallic and boron composite plates. One of the panels (#2) was unsymmetrically laminated with respect to the midsurface. The experimental buckling load correlated with the analytical results obtained using the reduced flexural stiffnesses in conjunction with the buckling analysis of Ashton and Waddoups.[7] Ashton[14] proposed an approximate solution method for the static response and buckling analyses based on an observation of the potential energy function. The approximation is exactly that of Reissner and Stavsky,[6] namely neglecting the coupling of the stress function and the transverse displacement in both the membrane strain compatibility equation and the transverse force equilibrium equation. However this analytical observation of the energy function[1] is improperly interpreted and therefore deserves attention.

Ashton[14] presents the following expression for the total potential energy§ of a general laminated plate:

$$V = \tfrac{1}{2}\iint[(e)^T(A)(e) + 2(e)^T(B)(\kappa) + (\kappa)^T(D)(\kappa)]dA \quad (1)$$

Then by introducing the constitutive equations for a general laminate,

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} e \\ \kappa \end{Bmatrix} \quad (2)$$

and the partial inversion scheme of Reissner and Stavsky,[6]

$$\begin{Bmatrix} e \\ M \end{Bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ BA^{-1} & D - BA^{-1}B \end{bmatrix} \begin{Bmatrix} N \\ \kappa \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ -B^{*T} & D^* \end{bmatrix} \begin{Bmatrix} N \\ \kappa \end{Bmatrix} \quad (3)$$

The energy expression of Eq. (1) becomes

$$V = \tfrac{1}{2}\iint[(N)^T(A^{-1})(N) + (\kappa)^T(D - BA^{-1}B)(\kappa)]dA \quad (4)$$

By observing that the curvature expressions are uncoupled from the stress resultants in Eq. (4), it was concluded that, "the coupling which will be involved in the governing differential equations for the problem, when written in terms of the stress resultants and plate curvatures, arises only through the compatibility condition."[14] It should be recognized that Eq. (4) is a potential energy formulation where the independent variables are the displacements, u, v, and w. Since the stress resultants are dependent variables in this formulation a proper application of the principle of minimum potential energy would lead to coupled equations of the form presented by Whitney.[11]

In order to develop a set of governing equations in terms of stress resultants and curvatures, one should follow the "generalized potential energy formulation" of Washizu[18] and generate a modifier Reissner energy formulation. Starting from the strain energy function of Eq. (1) and adding on the work terms and kinematics of deformation through Lagrange Multipliers which can be shown to be the stress resultants, the following generalized functional is obtained:

$$\begin{aligned} \Pi_1 = \iint\{&\tfrac{1}{2}[(e)^T(A)(e) + 2(e)^T(B)(\kappa) + (\kappa)^T(D)(\kappa)] - \\ &(e_x - u_{,x} - \tfrac{1}{2}w^2{}_{,x})N_x - (e_y - v_{,y} - \\ &\tfrac{1}{2}w^2{}_{,y})N_y - (e_{xy} - u_{,y} - v_{,x} - w_{,x}w_{,y})N_{xy} - \\ &(\kappa_x + w_{,xx})M_x - (\kappa_y + w_{,yy})M_y - \\ &(\kappa_{xy} + 2w_{,xy})M_{xy} - \bar{p}w\}dA - \\ &\{\textstyle\int(\bar{N}_x u + \bar{N}_{xy} v)dy \text{ or } \int[(u - \bar{u})N_x + \\ &(v - \bar{v})N_{xy}]dy\} - \{\textstyle\int(\bar{N}_{xy} u + \bar{N}_y v)dx \text{ or } \\ &\textstyle\int[(u - \bar{u})N_{xy} + (v - \bar{v})N_y]dx\} \end{aligned} \quad (5)$$

The independent variables of this expression are the strains, curvatures, stress resultants, moment resultants, and displacements. Regrouping the terms and integrating by parts leads to the following expression:

$$\begin{aligned} \Pi_1 = \iint\{&\tfrac{1}{2}[(e)^T(A)(e) + 2(e)^T(B)(\kappa) + \\ &(\kappa)^T(D)(\kappa) - 2(e)^T(N) - 2(\kappa)^T(M)] + \\ &\tfrac{1}{2}w^2{}_{,x}N_x + \tfrac{1}{2}w^2{}_{,y}N_y + w_{,x}w_{,y}N_{xy} - \\ &w_{,xx}M_x - w_{,yy}M_y - 2w_{,xy}M_{xy} - \bar{p}w - \\ &(N_{x,x} + N_{xy,y})u - (N_{xy,x} + N_{y,y})v\}dA + \\ &[-\textstyle\int(\bar{N}_x u + \bar{N}_{xy} v)dy \text{ or } +\int(\bar{u}N_x + \bar{v}N_{xy})dy] + \\ &[-\textstyle\int(\bar{N}_{xy} u + \bar{N}_y v)dy \text{ or } -\int(\bar{u}N_{xy} + \bar{v}N_y)dx] \end{aligned} \quad (6)$$

Using the partial inversion scheme of Reissner and Stavsky[6] to eliminate the membrane strains (e), and the resultant moments (M) leads to the following:

$$\begin{aligned} \Pi_1{}^* = \iint\{&-\tfrac{1}{2}[(N)^T(A)^{-1}(N) + \\ &(\kappa)^T(D - BA^{-1}B)(\kappa)] - \bar{p}w + \\ &(w_N)^T(N) + (w_\kappa)^T(B)(A)^{-1}(N) + \\ &(w_\kappa)^T(D - BA^{-1}B)(\kappa) - (N_{x,x} + N_{xy,y})u - \\ &(N_{xy,x} + N_{y,y})v\}dA + \text{boundary terms} \end{aligned} \quad (7)$$

where

$$(w_N)^T = \tfrac{1}{2}(w^2{}_{,x},\, w^2{}_{,y},\, 2w_{,x}\, w_{,y})^T \quad (8)$$

and

$$(w_\kappa)^T = -(w_{,xx},\, w_{,yy},\, 2w_{,xy})^T \quad (9)$$

Next, the curvatures are eliminated by utilizing the equations of kinematics of bending:

$$(\kappa)^T = (w_\kappa)^T \quad (10)$$

leading to

$$\begin{aligned} \Pi_1{}^* = \iint\{&-\tfrac{1}{2}[(N)^T(A)^{-1}(N) - 2(w_\kappa)^T(B)(A)^{-1}(N) - \\ &(w_\kappa)^T(D - BA^{-1}B)(w_\kappa)] - \bar{p}w + \\ &(w_N)^T(N) - (N_{x,x} + N_{xy,y})u - \\ &(N_{xy,x} + N_{y,y})v\}dA + \text{boundary terms} \end{aligned} \quad (11)$$

This then is the energy function from which the governing equation in terms of stress resultants and the transverse displacement should be developed. It is important to recognize that the coupling terms remain, simply as a consequence of the form of the constitutive equations. If the Airy stress

§ Whereas the expression presented is the potential energy for an unloaded plate, it can alternatively be viewed as the strain energy contribution to the total potential energy of a loaded plate.

function is used to represent the resultant stresses,

$$N_x = \partial^2\phi/\partial y^2 \quad N_y = \partial^2\phi/\partial x^2 \quad N_{xy} = -\partial^2\phi/\partial x\partial y \tag{12}$$

or

$$(N)^T = (\phi_{,yy}, \phi_{,xx}, -\phi_{xy})^T = (\phi)^T \tag{13}$$

The functional reduces to a modified Reissner energy form

$$\Pi_R^* = \iint\{-\tfrac{1}{2}[(\phi)^T(A)^{-1}(\phi) - 2(w_\kappa)^T(B)(A)^{-1}(\phi) - (w_\kappa)^T(D - BA^{-1}B)(w_\kappa)] - \bar{p}w + (w_N)^T(\phi)\}dA + \text{boundary terms} \tag{14}$$

where ϕ and w are the independent variables. The stationary condition for Π_R^* leads to two simultaneous partial differential equations of the form presented by Reissner and Stavsky.[6] One equation is a statement of membrane strain compatibility, whereas the other is a statement of transverse force equilibrium. The linear coupling terms which arise from the constitutive equations appear in both the equilibrium and compability equations. The elimination of these coupling terms from either equation is simply an approximation similar to those used by Donnell in developing the cylindrical shell equations or Reissner in developing the shallow spherical shell equations (see Ref. 19, for example). The ability of the approximate solution to adequately represent the exact solution has been demonstrated by Reissner and Stavsky,[6] Whitney and Leissa,[10] Ashton and Love,[13] Ashton[14] and Kicher.[20] However, it is meaningful to examine those configurations for which the approximate solution has enjoyed moderate success.

Cross Plied Plates

The governing equations for the transverse response of a cross plied plate can be expressed as follows[2]:

$$L_1w + L_4\phi = q_n \quad \text{(equilibrium)} \tag{15}$$

$$L_4w - L_2\phi = 0 \quad \text{(compatibility)} \tag{16}$$

where

$$L_1 = D_{11}^* \frac{\partial^4}{\partial x^4} + 2(D_{12}^* + 2D_{66}^*)\frac{\partial^4}{\partial x^2\partial y^2} + D_{22}^*\frac{\partial^4}{\partial y^4} \tag{17}$$

$$L_2 = A_{22}^* \frac{\partial^4}{\partial x^4} + (2A_{12}^* + A_{66}^*)\frac{\partial^4}{\partial x^2\partial y^2} + A_{11}^*\frac{\partial^4}{\partial y^4} \tag{18}$$

$$L_4 = B_{21}^* \frac{\partial^4}{\partial x^4} + (B_{11}^* + B_{22}^*)\frac{\partial^4}{\partial x^2\partial y^2} + B_{12}^*\frac{\partial^4}{\partial y^4} \tag{19}$$

The L_4 operator is the coupling term and the asterisks indicate the quantities are elements of the partially inverted constitutive equation, Eq. (3). For a cross plied plate of identical layers, it can be shown that[10]

$$A_{11} = A_{22}, B_{11} = -B_{22} \text{ and } D_{11} = D_{22}$$

Furthermore it can be shown that

$$A_{11}^* = A_{22}^* \quad D_{11}^* = D_{22}^* \quad B_{11}^* = -B_{22}^* \text{ and } B_{12}^* = -B_{21}^*$$

For this particular laminate the coupling operator reduces to[10]

$$L_4 = B_{21}^*[(\partial^4/\partial x^4) - (\partial^4/\partial y^4)] \tag{20}$$

Whitney and Leissa[11] have presented the exact solution to this set of equations for the case of simply supported rectangular plates. The stress function ϕ takes the form

$$\phi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{21}$$

where A_{mn} is found using Green's method[11] to satisfy free membrane boundary conditions. Substituting into Eq. (20), the algebraic magnitude of the L_4 operator can be determined:

$$L_4\phi = +B_{21}^* \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\pi^4 \times \left[\left(\frac{m}{a}\right)^4 - \left(\frac{n}{b}\right)^4\right] \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{22}$$

For the case of a square isotropic plate ($a = b$) subjected to a uniform pressure, $m = n = 1$ is the dominant term in the infinite series solution, showing an error of only $2\frac{1}{2}\%$ from the exact solution for the center point displacement.[21] If a similar dominance by the first term of the series were present in the cross plied plate, the single term solution would be uncoupled since $L_4\phi = 0$ for $m = n = 1$. As the aspect ratio of the plate increases, the error between the approximate solution and the exact solution will increase because of the presence of the coupling terms. Such an observation has been made by Whitney and Leissa[10] for the case of a free membrane boundary condition. However, for the case of a free normal-clamped tangential membrane boundary condition these observations do not hold. Further numerical studies are warranted.

Angle Ply Composites

The governing equations for the transverse response of an angle-plied composite plate can be expressed as [6,10]

$$L_1w - L_3\phi = \bar{p} \quad \text{(equilibrium)} \tag{23}$$

$$L_2\phi + L_3w = 0 \quad \text{(compatibility)} \tag{24}$$

where L_1 and L_2 are defined in Eqs. (17) and (18) and

$$L_3 = (B_{61}^* - 2B_{26}^*)\frac{\partial^4}{\partial x^3\partial y} + (B_{62}^* - 2B_{16}^*)\frac{\partial^4}{\partial x\partial y^3} \tag{25}$$

In this case, the L_3 operator introduces the coupling effects and again the asterisk quantities are elements of the matrix equation, Eq. (3). Consider the case of two identical layers at + and −45°. For this case $A_{11} = A_{22}$, $B_{16} = B_{26}$, and $D_{11} = D_{22}$. Furthermore it can be shown that

$$A_{11}^* = A_{22}^*, D_{11}^* = D_{22}^*, B_{16}^* = B_{26}^*, \text{ and } B_{61}^* = B_{62}^*$$

from which it can be shown that $B_{61}^* - 2B_{26}^* = B_{62}^* - 2B_{16}^*$. The L_3 operator can be expressed as

$$L_3 = (B_{61}^* - 2B_{26}^*)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{26}$$

Figure 6 is a plot of the coupling coefficients $(B_{61}^* - 2B_{26}^*)$ and $(B_{62}^* - 2B_{16}^*)$ for "Thornel"-50 in an epoxy matrix. The plot indicates that the coupling coefficients vanish at 0°, 90° and at some intermediate value near 45°. Although both coefficients do not vanish at 45° their values are small in comparison to the coupling coefficients at other angles. The error introduced in assuming zero values for these coupling terms in the vicinity of $40° < \theta < 50°$ is therefore minimized. Conversely the coupling coefficients assume a maximum value in the vicinity of $5° < \theta < 25°$ and $65° < \theta < 85°$. These regions should exhibit the maximum error in results when the coupling terms are neglected. This observation is confirmed by the results presented by Reissner and Stavsky[6] and Whitney and Leissa.[10]

Buckling of Unbalanced Plates

Consider next the bifurcation buckling analysis of a simply supported cross plied plate. Similar arguments can be advanced for the angle-ply plate if L_3 is substituted for L_4.

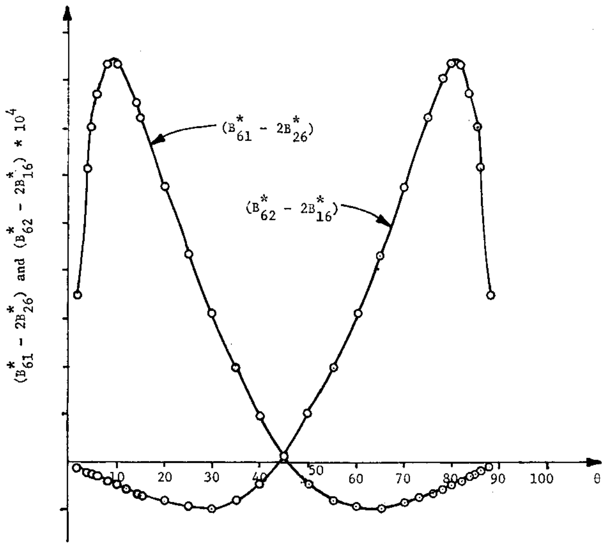

Fig. 6 (B_{61}^* − $2B_{26}^*$) and (B_{62}^* − $2B_{16}^*$) vs $\theta°$ for a Two-Ply "Thornel"-50 composite plate.

The nonlinear equations of equilibrium and compatibility are

$$L_1 w + L_4\phi = (N_x w_{,x} + N_{xy} w_{,y})_{,x} + (N_{xy} w_{,x} + N_y w_{,y})_{,y} \quad (27)$$

$$L_4 w - L_2\phi = w^2_{,xy} - w_{,xx} w_{,yy} \quad (28)$$

Expanding the equilibrium equation yields,

$$L_1 w - L_2\phi = (N_{x,x} + N_{xy,y}) w_{,x} + (N_{xy,x} + N_{y,y}) w_{,y}\, N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} \quad (29)$$

The first two terms of the right-hand side are exactly zero since the Airy stress function from which the stress resultants are calculated satisfies the membrane equilibrium equations. The governing equations reduce to

$$L_1 w + L_4\phi = N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} \quad (30)$$

$$L_4 w - L_2\phi = w^2_{,xy} - w_{,xx} w_{,yy} \quad (31)$$

Define each of the aforementioned variables with two components: prebuckling (o subscript) and buckling ($\tilde{\ }$),

$$w = w_o + \tilde{w} \quad \phi = \phi_o + \tilde{\phi} \quad N_x = N_{xo} + \tilde{N}_x \quad (32)$$

Substitute into the nonlinear differential equations,

$$L_1 w_o + L_1\tilde{w} + L_4\phi_o + L_4\tilde{\phi} = (N_{xo} + N_x)(w_{o,xx} + \tilde{w}_{,xx}) + \ldots \quad (33)$$

$$L_4 w_o + L_4\tilde{w} - L_{2o} - L_2\tilde{\phi} = (w_{o,xy} + w_{,xy})^2 - (w_{o,xx} + \tilde{w}_{,xx})(w_{o,yy} + \tilde{w}_{,yy}) \quad (34)$$

Examine the response prior to buckling,

$$L_1 w_o + L_4\phi_o = N_{xo} w_{o,xx} + 2N_{xyo} w_{o,xy} + N_{yo} w_{o,yy} \quad (35)$$

$$L_4 w_o - L_2\phi_o = w^2_{o,xy} - w_{o,xx} w_{o,yy} \quad (36)$$

Next restrict the prebuckling examination to the linear displacement regime (first stages of loading),

$$L_1 w_o + L_4\phi_o = 0 \quad (37)$$

$$L_4 w_o - L_2\phi_o = 0 \quad (38)$$

If the coupling term is neglected for this problem ($L_4 w_o = L_4\phi_o = 0$), the displacement response prior to buckling is pure membrane. Therefore the nonlinear terms will not enter until the plate buckles out of plane, i.e., $w_o = 0$, and the linear perturbation equations become

$$L_1\tilde{w} + L_4\tilde{\phi} = N_{xo}\tilde{w}_{,xx} + 2N_{xyo}\tilde{w}_{,xy} + N_{yo}\tilde{w}_{,yy} \quad (39)$$

$$L_4\tilde{w} - L_2\tilde{\phi} = \tilde{w}^2_{,xy} - \tilde{w}_{,xx}\tilde{w}_{,yy} \quad (40)$$

If the coupling terms are again neglected, $L_4\tilde{w} = L_4\tilde{\phi} = 0$, the equations reduce to the classical bifurcation equation with reduced flexural rigidities,

$$D_{xx}^* \frac{\partial^4\tilde{w}}{\partial x^4} + 2(D_{xy}^* + 2D_{ss}^*)\frac{\partial^4\tilde{w}}{\partial x^2\partial y^2} + D_{yy}^*\frac{\partial^4\tilde{w}}{\partial y^4} = N_{xo}\frac{\partial^4\tilde{w}}{\partial x^2} + 2N_{xyo}\frac{\partial^2\tilde{w}}{\partial x\partial y} + N_{yo}\frac{\partial^2\tilde{w}}{\partial y^2} \quad (41)$$

For the case of uniaxial loading on a plate with all four sides simply supported, the buckling load is expressed as

$$(N_x)_{cr} = \frac{\pi^2}{b^2}\left[D_{xx}^*\frac{m^2b^2}{a^2} + 2(D_{xy}^* + 2D_{ss}^*) + D_{yy}^*\frac{a^2}{m^2b^2}\right] \quad (42)$$

where m is the number of half waves in the x direction.

For the case of uniaxial loading on a plate with the unloaded edges free and the loaded edges simply supported, the buckling load is

$$(N_x)_{cr} = \pi^2 D_{xx}^*/a^2 \quad (43)$$

This represents, at best, a wide column approximation since it does not properly account for the moment and transverse shear conditions on the free edge.

If, however, the coupling terms are retained in the linear perturbation equation, Eq. (39), then the buckling load calculation is substantially more complicated. Whitney and Leissa[10] present results for the angle-ply composite under the assumption of a membrane prebuckling response. If the coupling terms are retained in the analysis prior to buckling, the transverse displacement may, in fact, not vanish. In this case, the nonlinear terms may enter into consideration, in which case buckling is not characterized as a bifurcation but rather an inflection point in the load-displacement diagram.

Indications of the prebuckling behavior can be obtained from physical observations. Consider the case of a cross ply composite subjected to an edge loading situation where the unloaded edges are free and the loaded edges are only required to remain planar (in plane displacements are not specified, whereas the transverse displacement can have a constant and a linear value). It is expected that such a plate would displace immediately upon application of the load similar to the case or an eccentric wide column including anticlastic curvatures. The addition of constraints to remove all transverse displacements on all of the edges (for example, simple support) would only inhibit the displacement. Therefore one might speculate that the cross ply composite suffers transverse displacement prior to buckling. Consider the case of an angle ply composite subjected to an edge loading, where the unloaded edges are free and the loaded edges are only required to remain planar. The plate would attempt to twist into a shape resembling a hyperbolic paraboloid upon application of a load. If all edges were then constrained to lie in a plane, this prebuckling mode of deformation would not be possible and one might speculate that the angle ply composite has a bifurcation point. The finite deflection results of Schmit and Monforton[23] support this observation and agree with the eigenvalue buckling analysis for angle ply plates presented by Whitney and Leissa.[10,12] A more rigorous buckling analysis is currently being pursued and the results are forthcoming.[24]

The consequences of neglecting the coupling is not simply an error in predicting the magnitudes as in the static response but rather a modification of the character of the response. When the coupling term is neglected the resulting governing equa-

Table 1 Theoretical and experimental buckling loads

Plate configuration	Material	Thickness in.	Critical load (lb/in.) 2 sides s. s. Theo.	2 sides s. s. Exp.	4 sides s. s. Theo.	4 sides s. s. Exp.
—	Aluminum	0.063	22.9	22.3	91.6	89.3
—	Aluminum	0.090	66.7	64.3	267	255
—	Steel	0.078	125	121	520	516
(90,+45,−45,90)	Fiberglass	0.103	15.4[b]	13.7	90.2[b]	120
(0,−45,+45,0)	Fiberglass	0.103	43.4[b]	36.6	90.2[b]	102
(45,90,90,−45)	Fiberglass	0.108	22.5	18.3	126	113
			18.9[a]		102[a]	
(−45,0,0,+45)	Fiberglass	0.108	26.8	23.0	126	108
(90,25,−25,90)	Fiberglass	0.103	15.4[b]	12.5	88[b]	101
(0,−65,65,0)	Fiberglass	0.103	42.6[b]	39.1	88[b]	88.4
(25,90,90,−25)	Fiberglass	0.085	20.5	12.8	64	57.3
			16.6[a]		52.9[a]	
(−65,0,0,65)	Fiberglass	0.085	11.2	9.51	64	68.8
			10.9		52.9[a]	
4(0), 4(90)	Fiberglass	0.090	23.9	15.4	69.5	63.7
			17.6[a]		55.4[a]	
4(90), 4(0)	Fiberglass	0.090	23.9	15.3	69.5	72.7
			17.6		55.4[a]	
(0,90,90,0)	"Thornel"-25	0.055	12.5	11.3	19.1	21.7
9(0,90)	"Thornel"-25	0.121	105	84.9	204	189
9(±45)	"Thornel"-25	0.116	44.9	33.8	292	313
9(±45)	"Thornel"-25	0.116	44.9	34.7	292	283
5(0,90)	"Thornel"-40	0.043	12.7	10.8	18.7	15.5
5(90,0)	"Thornel"-40	0.043	3.83	2.74	18.7	16.3
(0,90,90,0)	"Thornel"-40	0.034	6.92	5.26	8.76	6.69
(90,0,0,90)	"Thornel"-40	0.034	1.29	0.848	7.44	6.65
(0,45,45,45,0)	"Thornel"-40	0.038	9.30	8.27	12.7	13.8
4(0), 4(90)	"Thornel"-40	0.091	62.3	20.7	142	72.1
			23.7[a]		74.2[a]	
4(90), 4(0)	"Thornel"-40	0.091	62.3	19.7	142	73.7
			23.7[a]		74.2[a]	
4(+25), 4(−25)	"Thornel"-50	0.072	485	17.5	118.4	41.0
			176[a]		42.9*	
4(+65), 4(−65)	"Thornel"-50	0.072	50.7	33.2		
			33.8[a]			
4(+45), 4(−45)	"Thornel"-50	0.076	363	10.5	264.1	71.0
			132[a]		82.7[a]	
4(−45), 4(+45)	"Thornel"-50	0.076	363	11.2	264.1	75.8
			132[a]		82.7[a]	
4(0), 4(90)	"Thornel"-50	0.074	950	22.5	212.2	59.6
			312[a]		80.0[a]	
4(90), 4(0)	"Thornel"-50	0.074	950	25.8	212.2	60.3
			312[a]		80.0[a]	
4(0), 4(45)	"Thornel"-50	0.090			404.9	120.9
					177.0[a]	
4(90), 4(−45)	"Thornel"-50	0.090			404.9	124.9
					177.0[a]	

Note: All theoretical buckling loads are determined from classical orthotropic theory unless otherwise noted.

[a] Reduced flexural stiffness method.
[b] Coupling terms neglected, angle plies adjacent to reference surface.

tions reduce to an eigenvalue analysis providing, at best, an upper bound to the nonlinear response. When the coupling terms are included, the response is not simply pure membrane bifurcating into a bending displacement but rather a totally nonlinear behavior for which buckling is defined by the stationary conditions of the potential energy.

There are two special cases of unbalanced laminates for which a bifurcation analysis might yield meaningful results. Consider first the square simply supported plate cross laminated from two identical plies. The static transverse response was shown to be exactly uncoupled for deformation in the first mode ($m = n = 1$). Therefore the first mode is a bifurcation mode of buckling if the nonlinear prebuckling analysis is ignored. Although this plate may bend due to the edge loading prior to buckling, none of the bending is in the first mode. At the point of buckling the plate can therefore change from one bent mode into another as has been observed for initially imperfect columns.[22] Since the first mode dominates the prebuckling response, the buckling load and the experimental behavior will be closely approximated by the bifurcation analysis.

Next consider an angle-ply laminate of two identical plies oriented at ±45°. It was shown earlier that the effects of the coupling terms on the response of a simply supported angle-ply plate were minimized at $\theta = \pm 45°$. Therefore the reduced flexural stiffness method should provide an eigenvalue buckling load near the exact solution. The results of the experimental program reported herein indicate the usefulness of the reduced flexural stiffness method for predicting the buckling loads of both cross-ply and angle-ply composites.

Experimental Results

Table 1 contains the results of buckling tests on aluminum, steel, fiberglass, and "Thornel" plates. The column indicated as plate configuration specifies the fiber orientations in laminated plates. The following conventions have been used: a) (0,90,90,0) is a four-layer plate with outside plies at

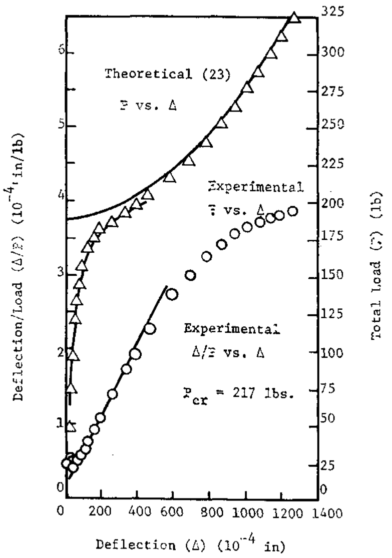

Fig. 7 Southwell plot and load deflection curve for (0, 90, 90, 0) "Thornel"-25 plate, 4-sides supported.

0° and both interior plies at 90°, b) 9(0,90) is a nine-layer plate with adjacent plies at right angles, and c) 4(0), 4(90) is an eight layered plate; on one side of the midsurface of the plate all plies are at 0°, on the other side they are at 90°.

Experimental and analytical results for the buckling of plates with two sides supported as well as four sides supported are presented. An analytical prediction based on classical theory is presented for each case, using Eqs. (42) and (43) with the flexural stiffnesses defined by Eq. (2). For the unbalanced plates, a classical solution based on reduced flexural stiffnesses is also presented, if the coupling effects result in a significant change in the buckling load.

The experimental buckling load is determined from the load vs transverse displacement data using a Southwell plot. Figure 7 is a load displacement plot and a Southwell plot for the "Thornel"-25 (0,90,90,0) plate with all four sides simply supported. It is important to recognize that the Southwell plot technique is based on the observation that the true load displacement curve is a hyperbolic shaped asymptote to the horizontal eigenvalue. Therefore the Southwell plot will have a linear region from which to deduce a buckling load, only during the early stages of loading. Note also that the Southwell plot is plagued with sporadic data at the initial onset of loading due to fixture misalignment. For extremely thin plates, the linear region of the Southwell plot is severely limited and an additional source of error is introduced.

Discussion

The metallic plates exhibited good agreement of analysis with experiment. Eight of the fiberglass specimens had the membrane-twisting coupling produced by $a \pm \theta$ lamination, but the full effect of such coupling was diminished by the presence of the 0° ply. These four-ply plates were selected in pairs to demonstrate the effects of placing the $\pm\theta$ adjacent to the reference surface or at the extreme position from the reference surface. Two-fiberglass specimens were laminated to produce a cross plied plate with coupling between membrane and bending. Because of the low orthotropicity ratio, E_L/E_T none of the fiberglass specimens exhibited large deviations from classical theory due to coupling effects.

The "Thornel"-25 series of plates were tested early in the program and consequently all were laminated in a balanced configuration. The correlation of analysis with experiment is well within experimental error for this series of plates. Four of the "Thornel"-40 specimens were laminated in a balanced configuration. One "Thornel"-40 specimen was balanced laminate but the plies were oriented to produce in-plane shear coupling. The effects of in-plane direct stress-shear stress coupling have been ignored in calculating the buckling loads[7,8] but the effects appear to be less than that due to bending-membrane coupling for the class of plates studied. Two "Thornel"-40 specimens were cross plied and the bending membrane coupling severely affected the "buckling load." Correlation of the experimental data with a buckling load predicted using the reduced flexural stiffnesses for this plate was remarkably good.

The "Thornel"-50 series of plates all display some form of bending membrane coupling, which in some cases is shown to reduce the buckling load by a factor of three. The correlation between experimental results and analytical results is not as good however due to the poor fabrication quality. The result of this series of plates does however confirm the trend of reduced buckling loads for elastically unbalanced plates.

The experimentally determined buckling loads for all specimens simply supported on the loaded edges and free on the sides were below the wide column analytical result. This was caused by the artificial constraint in the approximate solution requiring the transverse displacement to be a function only of x at the expense of violating the free edge boundary conditions. The buckling loads for the metallic plates with all four sides simply supported indicate the possible presence of biaxial load due to friction in the loading fixture,[13] although the maximum error observed was only 6%. The four-ply fiberglass specimens with the angle plies adjacent to reference surface all had buckling loads exceeding the analytical prediction with the maximum error for the case where the outside plies are perpendicular to the load. On the other hand when the angle plies are placed on the outside, the buckling loads tended to fall below the analytical predictions. These observations are specifically noted for the $\pm 45°$ case where the coupling terms assume minimum values. Results from all four-ply (and five-ply "Thornel") plates indicate that some problems exist in determining the exact properties of very thin plates. For example, the surfaces of two of the plates were covered with tiny ridges of matrix material left by the release cloth used in the curing process. If a standard flat micrometer was used to determine the thickness, the resulting error in the buckling load prediction resulting from the added thickness was as high as 37% in one case.

Although the "Thornel"-50 plates did not generally exhibit good correlation of experiment with theory, the effects of in-plane shear coupling for highly anisotropic plates remain unanswered.

Two additional sources of error should be noted. If the wedges on the loaded edges are not free to rotate due to minor misalignment then an edge fixity is introduced into the experiment raising the apparent buckling load. Since the unbalanced plates had to be cured in balanced sublaminates and subsequently bonded with a room temperature epoxy, estimates of the bond thickness and consequently the location of the plies from the reference had to be made. Minor errors in the bond thickness estimate can have a major effect on the calculations of both the coupling terms (B) and the flexural rigidities (D).

The effects of initial imperfections can be seen from the plot in Fig. 7. The theoretical load versus deflection curve was determined by Schmit and Monforton[23] using a nonlinear finite element computer capability. The true response which includes the effects of initial imperfections follows the theoretical curve as an asymptote. The Southwell plot for this case gives a critical load of 217 lb, whereas the theoretical buckling load was 196 lb. Classical buckling theory for this case gives 191 lb.

Conclusions

All of the theoretical methods considered appeared to give accurate buckling load predictions for those plates to which they were meant to apply. The classical orthotropic theory gave accurate results as long as the bending-membrane coupling terms were small, but the theory was very nonconservative if the coupling terms were large. For those cases where the bending-membrane coupling terms are substantial, an upper bound on the "buckling load" can be determined from classical buckling theory using reduced flexural stiffnesses.

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